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Optimal Modal-Space Control of Flexible Gyroscopic Systems

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A solution for the optimal control of large-order gyroscopic systems using quadratic performance index is presented. The approach is based on independent modal-space control, and it requires the solution of $n/2$ decoupled 2×2 matrix Riccati equations (one for each pair of conjugate modes) instead of a general $n \times n$ matrix Riccati equation, where n is the number of modes to be controlled. The solution of the 2×2 steady-state matrix Riccati equations can be obtained in closed-form. Moreover, the transient solution is obtained by using augmented matrix formulation for 2×2 matrices, and it reduces to the inversion of such matrices, a very simple operation. The solutions obtained via the modal approach exhibit dependence of the control gains on the system natural frequencies, thus providing physical insight into the system behavior. This optimal modal-space control approach becomes increasingly attractive as the order of the system increases. Because the equations of motion are the result of spatial discretization of a distributed-parameter system, the relation between the spatial distribution of actuators and the (time) optimal control forces for the discretized system is of great interest; it receives special consideration here. The method is applied to a dual-spin flexible spacecraft.

Introduction

THE interaction between structural flexibility and control of large flexible spacecraft has become an important problem in space technology, and has received a great deal of attention recently.[1] A flexible spacecraft represents a distributed-parameter system, which in theory has an infinite number of degrees of freedom. In practice, the system must be discretized. Hence, the control of flexible spacecraft poses serious computational and dynamic modelling problems.

The control theory for discrete dynamical systems is relatively well developed. For the most part, however, general theories and their applications do not take into account many physical properties of the system under consideration, so that much physical insight and computational efficiency is often lost. Hence, in many cases, it is advisable to abandon generalities, and take advantage of given system properties to develop a control theory that is general for this more restricted class of systems, thus gaining efficiency and insight. One large class of problems for which this approach is particularly fruitful is that of gyroscopic systems. The control of flexible gyroscopic systems is not only an interesting problem, but also one of strong current interest.

It has been shown that control schemes taking advantage of the special dynamic features of the system lead to considerable computational efficiency in the control of systems with a large number of degrees of freedom. A control system designed especially for large gyroscopic systems seems to have been proposed only in Refs. 2-5. The approach used in these references is that of designing the control in modal-space, whereby control loops are designed by employing decoupled observers. This approach is not to be confused with modal control (see, for example, Ref. 6) in which control is designed in the actual space. Modal-space control has been applied to a spinning spacecraft[2] and a dual-spin spacecraft.[3] Reference 4 unifies the methodology for distributed systems, and Ref. 5 advances the method by considering problems of actual control implementation. In this latter reference, the distribution of controllers and their control time histories are specified from a synthesis of modal control laws for the discrete (in space) system dynamics. Another basic concept advanced in Refs. 2-5 is that of independent mode control, whereby the response of one mode is not affected by the response of the other modes. The control laws in modal-space used in Refs. 2-5, namely on-off and proportional controls, were not necessarily optimal. Moreover, no attempt was made in these references to optimize the control.

Optimal control using quadratic performance measure is well known.[7,8] A solution of the Riccati equation, or its discrete (in time) counterpart, allows the computation of feedback control matrices in advance for control implementation. However, as the order of the system increases, the computational time increases as the square of the order of the system, and in the process numerical difficulties are often encountered.[8] Such difficulties are sometimes insurmountable, so that this approach becomes unfeasible for large-order systems (>40). For example, for large-order systems, the approach based on Potter's method,[9] or variations of this method[8] for either the transient or the steady-state solution of the matrix Riccati equation, involves inversion of large-order matrices. In addition to numerical difficulties inherent in the inversion of large-order matrices, the computational time required, especially for a transient solution, is sufficiently large to render such methods unfeasible as the order of the system increases. In the case of linear gyroscopic systems, these difficulties are not inherent in the system, and often arise because one does not take advantage of the special nature of the system to decouple the equations governing the system behavior. Such decoupling is a relatively simple task for linear gyroscopic systems.[10,11]

Modal State Vector Formulation

The motion of flexible spacecraft can be described by a set of coordinates, some depending only on time and the rest depending on both time and space. Accordingly, some of the equations of motion for such a system are ordinary differential equations (in time) and the rest are partial differential equations. Such equations are being termed as hybrid. The behavior of a flexible spacecraft disturbed from the equilibrium position of uniform spin and that of a dual-spin spacecraft with a flexible despun section are perfect examples of dynamic systems described by hybrid sets of

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equations. The rigid-body rotations $\theta_i(t)$ from the equilibrium position of a system of spacecraft axes represent coordinates depending on time alone. The elastic displacement $u(P,t)$ of a given flexible member of the spacecraft relative to this system of axes represents a coordinate dependent on both space and time.

Sets of hybrid differential equations are customarily solved by discretizing in space by such methods as the finite-element technique, the assumed modes method, or the lumped parameter method.[12] In particular, according to the assumed modes method, one can represent the elastic displacement vector by a linear combination of space-dependent admissible vector functions ϕ_i multiplied by time-dependent generalized coordinates ζ_i in the form

$$u(P,t)=\sum_{i=1}^{n-3}\phi_i(P)\zeta_i(t)=\Phi(P)\zeta(t) \tag{1}$$

where P denotes spatial position, $\Phi(P)$ is the matrix of the admissible vector functions, and ζ is the generalized elastic coordinate vector. Assuming that the rotational and elastic disturbances are small, we introduce the n-dimensional configuration vector

$$q(t)=[\theta^T(t) \mid \zeta^T(t)]^T \tag{2}$$

Then, it can be shown that the equations for small motions of the system about the equilibrium position reduces to (see, for example, Ref. 13)

$$m\ddot{q}+g\dot{q}+kq=Q \tag{3}$$

where

$$m=m^T \qquad g=-g^T \qquad k=k^T \tag{4}$$

The matrices m, g, and k are mass, gyroscopic, and stiffness matrices, respectively. The matrix g includes the Coriolis effects, and the matrix k includes the elastic and centrifugal effects. The matrix m is positive definite, and the matrix k is assumed to be positive definite. Equation (3) merely represents the discretized-in-space version of the distributed-parameter system originally described by the hybrid equations of motion. The vector Q is the generalized force vector of the discretized system dynamics, and should not be confused with the actual force vector.

The solution of Eq. (3) can be conveniently obtained by transforming it into a first-order vector equation. To this end, we introduce the $2n$-dimensional state vector $x(t)$ and associated force vector $X(t)$ defined by

$$x=[\dot{q}^T \mid q^T]^T \qquad X=[Q^T \mid 0^T]^T \tag{5}$$

as well as the $2n\times 2n$ matrices

$$M=M^T=\left[\begin{array}{c|c} m & 0 \\ \hline 0 & k \end{array}\right] \qquad G=-G^T=\left[\begin{array}{c|c} g & k \\ \hline -k & 0 \end{array}\right] \tag{6}$$

Using Eqs. (5) and (6), Eq. (3) can be rewritten in the form of the state equation

$$M\dot{x}(t)+Gx(t)=X(t) \tag{7}$$

The eigenvalue problem associated with Eq. (7) has the form

$$(\lambda_r M+G)x_r=0 \qquad (r=1,2,\ldots,2n) \tag{8}$$

where λ_r and x_r are a constant scalar and constant vector, respectively. It can be shown that the eigenvalues are pure imaginary complex conjugates, and the eigenvectors are also complex conjugates.[10] Denoting the eigenvalues by $\lambda_r=i\omega_r$, $\bar{\lambda}_r=-i\omega_r$ and the eigenvectors by $x_r=y+iz_r$, $\bar{x}_r=y_r-iz_r$ $(r=1,\ldots,n)$ where ω_r are the spacecraft natural frequencies, it can be shown that the eigenvectors are orthogonal with respect to the matrix M and in a certain sense with respect to the matrix G.[11] Forming the modal matrix P in the form

$$P=[y_1z_1y_2z_2\ldots y_nz_n] \tag{9}$$

we can write

$$P^TMP=I \tag{10}$$

where I is the $2n\times 2n$ unit matrix. It also follows that

$$-P^TGP=A=\text{block-diag } A_r=\text{block-diag}\begin{bmatrix} 0 & \omega_r \\ -\omega_r & 0 \end{bmatrix} \tag{11}$$

where A is a block-diagonal matrix with 2×2 matrices A_r on the main diagonal. The fact that a similarity transformation using the modal matrix P reduces the matrix M to the identity matrix, and the matrix G to the block-diagonal matrix A enables us to uncouple the system into n pairs of second-order differential equations as shown in the following section.

Let us consider the linear transformation

$$x=\sum_{r=1}^{n}[\xi_r(t)y_r+\eta_r(t)z_r]=Pw \tag{12}$$

where

$$w=[\xi_1\eta_1\xi_2\eta_2\ldots\xi_n\eta_n]^T \tag{13}$$

is the modal state vector with the components in the form of pairs of conjugate generalized modal coordinates. Introducing Eq. (12) into Eq. (7) and using Eqs. (10) and (11) we obtain

$$\dot{w}=Aw+W \tag{14}$$

where

$$W=P^TX \tag{15}$$

is the associated modal force vector. Equation (14) represents a set of n pairs of first-order differential equations in the generalized conjugate coordinates $\xi_r(t)$ and $\eta_r(t)$.

For future reference, let us write the Hamiltonian of the open-loop gyroscopic system. The definition of the Hamiltonian is (see, for example, Ref. 14).

$$\mathcal{H}=\sum_{k=1}^{n}\frac{\partial L}{\partial\dot{q}_k}\dot{q}_k-L=\tfrac{1}{2}\dot{q}^Tm\dot{q}^T+\tfrac{1}{2}q^Tkq=\tfrac{1}{2}x^TMx \tag{16}$$

so that, recalling Eqs. (10) and (12), we can further write

$$\mathcal{H}=\tfrac{1}{2}wP^TMPw=\tfrac{1}{2}w^Tw=\tfrac{1}{2}\|w\|^2 \tag{17}$$

Hence, the Hamiltonian of the system is simply one half of the Euclidean norm of the modal state vector w.

At this point, it is perhaps appropriate to pause and reflect on the order of the dynamic simulation. An exact description of a flexible spacecraft requires that the order be infinite. In practice, it is necessary to truncate the system, but for an accurate simulation of the system the order $2n$ can still be very large, e.g., $n>100$. Most control algorithms have yet to prove their adequacy for such high-order systems. In many cases, however, the difficulty is not inherent in the system, but is a direct result of the approach used, namely working with the coupled system. Indeed, in this paper we propose a method which takes advantage of the system properties to decouple the $2n$ simultaneous first-order equations into n independent pairs of first-order modal equations. Optimal control of second-order systems can be effected without difficulty. In essence, by transforming to modal coordinates and using a special form of controls, the limitation on the order of a linear gyroscopic system that can be controlled optimally has been shifted from the ability to solve high-order Riccati equations to the ability to solve high-order eigenvalue problems for real symmetric matrices. It is clear that the capability of solving eigenvalue problems for real symmetric matrices is ample and exceeds by several orders of magnitude the capability of solving Riccati equations.

Design of Control in Modal Space

Considering Eqs. (15) and (10), the generalized control vector can be written in the form

$$X=(P^T)^{-1}W=MPW \tag{18}$$

It will be convenient to partition the matrix P and the modal control vector W as follows

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad W=[W_C^T \; W_R^T]^T \tag{19}$$

where P_{ij} $(i,j=1,2)$ are $n\times n$ matrices, and W_C and W_R are n-vectors. Inserting Eqs. (19) into Eq. (18), we obtain

$$Q=m(P_{11}W_C+P_{12}W_R) \tag{20a}$$

$$0=k(P_{21}W_C+P_{22}W_R) \tag{20b}$$

Equation (20b) can be regarded as a constraint equation to be satisfied by the control vector W. The equation can be used to write

$$W_R=-P_{22}^{-1}P_{21}W_C \tag{21}$$

where we note that P_{22} is nonsingular by the very nature of P. Substituting Eq. (21) into Eq. (20a), the generalized control vector becomes

$$Q=m(P_{11}-P_{12}P_{22}^{-1}P_{21})W_C \tag{22}$$

Equation (22) establishes a unique relation between the generalized control vector Q and the control vector W_C. Introducing the notation

$$A=\begin{bmatrix} A_C & 0 \\ 0 & A_R \end{bmatrix} \tag{23}$$

where A_C and A_R are $n\times n$ matrices, as well as the notation

$$w=[w_C^T \; w_R^T]^T \tag{24}$$

we can rewrite Eq. (14) in the form

$$\dot{w}_C=A_Cw_C+W_C \qquad \dot{w}_R=A_Rw_R+W_R \tag{25}$$

The vector W_C can be regarded as arbitrary and chosen so that the modal vector w_C can be controlled in a prescribed manner. Hence, the vector w_C can be identified as being associated with the controlled modes, and the vector w_R with the uncontrolled (residual) modes. Because of the special nature of the matrix A, n will be taken as an even integer.

The control vector W_C can be taken as a linear function of the state vector w, so that by Eq. (21) the vector W_R will also be a linear function of the state vector w. This type of control is referred to as proportional control. For proportional control we can write the relation between the control vector and the state vector in the form

$$W=Fw \tag{26}$$

where F is the modal control gain matrix. We shall partition the gain matrix as follows:

$$\begin{bmatrix} W_C \\ W_R \end{bmatrix}=\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} w_C \\ w_R \end{bmatrix} \tag{27}$$

or

$$W_C=F_{11}w_C+F_{12}w_R \qquad W_R=F_{21}w_C+F_{22}w_R \tag{28}$$

Inserting Eqs. (28) into the constraint equation, Eq. (20b), and equating the coefficient matrices of w_C and w_R, we obtain

$$F_{21}=-P_{22}^{-1}P_{21}F_{11} \tag{29a}$$

$$F_{22}=-P_{22}^{-1}P_{21}F_{12} \tag{29b}$$

We are interested in independent control of the modes, so that the modal control force associated with each pair of controlled conjugate generalized modal coordinate must depend only on these coordinates, i.e.,

$$W_r=W_r(w_r) \qquad (r=1,2,\ldots,n/2) \tag{30}$$

where $w_r=[\xi_r,\eta_r]^T$ and $W_r=[W_{\xi r},W_{\eta r}]^T (r=1,2,\ldots,n/2)$. Hence, for independent control of modes, F_{12} must be chosen as zero, so that from Eq. (29b) it follows that F_{22} is also zero, and F_{11} must be a block-diagonal matrix. The nonzero elements of the matrix F_{11} can be chosen such that the vector w_C can be controlled in a prescribed manner, but the choice is not unique. For example, these elements can be chosen to satisfy certain additional requirements rendering the control "optimal." References 2-5 have applied the independent modal-space control laws described previously to gyroscopic systems, such as single- and dual-spin spacecraft, but none of the controls were optimal in any sense. In the next section, optimal control laws are given for a quadratic performance criterion.

Independent Modal-Space Optimization

It was stated in the previous section that independent control of modes is achieved if the control forces associated with each pair of controlled conjugate generalized modal coordinates depend only on these coordinates. This guarantees complete decoupling of the modes. In view of this, we can write a quadratic performance criterion for each pair of ξ_r and η_r to be controlled in the state vector w_C in the form

$$J_r=\frac{1}{2}\begin{bmatrix} \xi_r-\hat{\xi}_r \\ \eta_r-\hat{\eta}_r \end{bmatrix}^T\begin{bmatrix} H_{\xi r} & 0 \\ 0 & H_{\eta r} \end{bmatrix}\begin{bmatrix} \xi_r-\hat{\xi}_r \\ \eta_r-\hat{\eta}_r \end{bmatrix}$$
$$+\frac{1}{2}\int_{t_0}^{t_f}\left\{\begin{bmatrix} \xi_r \\ \eta_r \end{bmatrix}^T\begin{bmatrix} Q_{\xi r} & 0 \\ 0 & Q_{\eta r} \end{bmatrix}\begin{bmatrix} \xi_r \\ \eta_r \end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} W_{\xi r} \\ W_{\eta r} \end{bmatrix}^T\begin{bmatrix} R_{\xi r} & 0 \\ 0 & R_{\eta r} \end{bmatrix}\begin{bmatrix} W_{\xi r} \\ w_{\eta r} \end{bmatrix}\right\}dt \tag{31}$$

where $W_{\xi r}$ and $W_{\eta r}$ are the control forces on the dynamics corresponding to the coordinates ξ_r and η_r. The symbols $\hat{\xi}_r$ and $\hat{\eta}_r$ denote reference values to which the coordinates ξ_r and $\hat{\eta}_r$ are to be driven. The case in which these reference values are zero is known as the regulator problem. In the sequel, we shall only be concerned with the regulator problem, as the results can be easily generalized to the case in which $\hat{\xi}_r, \hat{\eta}_r \neq 0$. Because every pair ξ_r, η_r is controlled independently of any other pair, for all of the coordinates to be controlled, we can write the total quadratic cost as

$$J_C = \sum_{r=1}^{n/2} J_r \tag{32}$$

Hence, J_C will be minimized if each term J_r is minimized, and the minimization of each term J_r can be accomplished independently. To this end, we can write the modal performance index

$$J_r = \frac{1}{2} w_r^T(t_f) H_r w_r(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [w_r^T(t) Q_r w_r(t) + W_r(t)^T R_r W_r(t)]\,\mathrm{d}t \tag{33a}$$

where

$$H_r = \begin{bmatrix} H_{\xi r} & 0 \\ 0 & H_{\eta r} \end{bmatrix} \qquad Q_r = \begin{bmatrix} Q_{\xi r} & 0 \\ 0 & Q_{\eta r} \end{bmatrix}$$

$$R_r = \begin{bmatrix} R_{\xi r} & 0 \\ 0 & R_{\eta r} \end{bmatrix} \tag{33b}$$

The vector w_r must satisfy

$$\dot{w}_r = A_r w_r + W_r \tag{34}$$

The final time t_f is assumed to be fixed. Moreover, H_r and Q_r are real symmetric, positive semidefinite matrices, and R_r is a positive definite matrix. We shall assume that the controls are not bounded and $w_r(t_f)$ is free. Comparing the first term of the integrand in J_r with the open-loop Hamiltonian, Eq. (16), we decide to choose the matrix Q_r equal to the 2×2 unit matrix. This permits us to interpret the minimization of the performance index J_r as the process of keeping the modal state vector as close to the origin of the modal state space as possible without too much control effort, and without increasing the Hamiltonian of the open-loop gyroscopic system. Equations (33) and (34) represent a classical optimal, linear regulator problem.

The minimization of the control performance can suitably be summarized by defining the Hamiltonian function $\mathcal{H}_r^c$ of the controlled, closed-loop dynamical system

$$\mathcal{H}_r^c(w_r(t), W_r(t), p_r(t)) = a_r(w_r(t), W_r(t), t) + p_r^T(t) h_r(w_r(t), W_r(t), t) \tag{35}$$

where

$$a_r(w_r(t), W_r(t), t) = \tfrac{1}{2}\{w_r^T(t) Q_r w_r(t) + W_r(t)^T R_r W_r(t)\} \tag{36a}$$

$$h_r(w_r(t), W_r(t), t) = A_r w_r(t) + W_r(t) \tag{36b}$$

and $p_r(t)$ is a two-dimensional vector of Lagrange multipliers known as the costate vector. The necessary conditions for minimization are given by (see, for example, Ref. 7)

$$\dot{w}_r^*(t) = \frac{\partial \mathcal{H}_r^c}{\partial p_r} \qquad \dot{p}_r^*(t) = -\frac{\partial \mathcal{H}_r^c}{\partial w_r} \qquad 0 = \frac{\partial \mathcal{H}_r^c}{\partial W_r}$$

$$(t_0 < t < t_f) \tag{37}$$

with the boundary conditions

$$w_r^*(t_0) = w_{r0} \qquad p_r^*(t_f) = H_r w_r^*(t_f) \tag{38}$$

An asterisk denotes final optimal quantities. Inserting Eq. (35) into Eqs. (37), we obtain

$$\dot{w}_r^*(t) = A_r w_r^*(t) + W_r^*(t) \tag{39a}$$

$$\dot{p}_r^*(t) = -Q_r w_r^*(t) - A_r^T p_r^*(t) \tag{39b}$$

$$0 = R_r W_r^*(t) + p_r^*(t) \tag{39c}$$

Equation (39c) gives the optimal control as

$$W_r^*(t) = -R_r^{-1} p_r^*(t) \tag{40}$$

where $p_r^*(t)$ can be obtained by considering the solution of Eqs. (39a, b). This solution can be written in the form

$$p_r^*(t) = K_r(t) w_r^*(t) \tag{41}$$

where the 2×2 matrix $K_r(t)$ satisfies

$$\dot{K}_r(t) = -K_r A_r - A_r^T K_r - Q_r + K_r R_r^{-1} K_r \tag{42}$$

which is subject to the end condition

$$K_r(t_f) = H_r \tag{43}$$

Equation (42) is known as the Riccati equation. It is not difficult to show that the matrix K_r is symmetric.[7] Using Eqs. (40) and (41), the optimal control becomes

$$W_r^*(t) = -R_r^{-1} K_r(t) w_r^*(t) = f_r^*(t) w_r^*(t) \quad (r = 1,2,\ldots,n/2) \tag{44}$$

where

$$f_r^*(t) = -R_r^{-1} K_r(t) \qquad (r = 1,2,\ldots,n/2) \tag{45}$$

is an optimal feedback gain matrix.

The determination of the control gain for each pair ξ_r and η_r has now been reduced to obtaining the solution of the 2×2 classical matrix Riccati equation, Eq. (42). Noting that the control input on the controlled modes W_C has the form

$$W_C = [W_1^T W_2^T \ldots W_r^T \ldots W_{n/2}^T]^T \tag{46}$$

we can immediately deduce that the optimal control of a large-order dynamic system has been reduced to the solution of $n/2$ second-order Riccati equations, one for each pair of system modes. This computationally attractive result has been achieved by the decoupling procedure leading to independent modal control.

Recalling that W_r is the control vector for the pair ξ_r, η_r, and using Eqs. (44) and (46) we can finally write

$$W_C^* = F_{11}^* w_C^* \tag{47}$$

in which

$$F_{11}^* = \text{block-diag}\,[f_r^*] \tag{48}$$

where the 2×2 matrices $f_r^*(r=1,2,...,n/2)$ are given by Eq. (45). Finally, using Eqs. (22) and (47), we can write the optimal control input Q^* on the coupled dynamics

$$Q^*=m(P_{11}-P_{12}P_{22}^{-1}P_{21})W_C^*=m(P_{11}-P_{12}P_{22}^{-1}P_{21})F_{11}^*w_C^* \tag{49}$$

which assumes that w_C^* vector is available for control implementation. The vector w_C^* can be estimated by using an observer in the control loop. Details of the design of a modal state estimator have been presented in Ref. 5. It suffices to say here that the control law given by Eq. (49) is physically realizable.

We shall be interested in a special type of control, namely one in which $W_{\xi r}$ is zero, and $W_{\eta r}$ alone controls the behavior of the conjugate pair ξ_r and η_r. This type of control is not without physical foundation. Indeed, because the equations for ξ_r and η_r are coupled, in controlling η_r one also controls ξ_r. The desired effect can be achieved by simply taking $R_{\xi r}$ to be infinite, in which case all the terms associated with $R_{\xi r}^{-1}$ in Eq. (44) vanish. As a result, the first row of the matrix f_r^*, Eq. (45), reduces to zero, and the solution of Riccati's equations simplifies considerably.

Next, we shall consider several aspects of Riccati's equation:

A. Steady-State Solution

Kalman has shown[15] that if the system is completely controllable and $H_r=0$ and A_r, Q_r and R_r are constant matrices, then $K_r(t)\rightarrow K_r=\text{const}$ as $t_f\rightarrow\infty$. In this case, the solution of Eq. (42) is known as the steady-state solution and can be obtained readily by setting $\dot{K}_r=0$ in Eq. (42). Recalling the form of A_r from Eq. (11) and that $R_{\xi r}^{-1}=0$, the steady-state Riccati equation reduces to

$$\omega_r(JK_r-K_rJ)+(1/R_{\eta r})K_rHK_r=Q_r=I_2 \tag{50}$$

where I_2 is the 2×2 unit matrix and

$$J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \qquad H=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

Equation (50) represents three nonlinear algebraic equations with the entries K_{11r}, K_{22r}, and $K_{12r}=K_{21r}$ of the matrix K_r as the unknowns. It can be shown that the solution of the algebraic equations is

$$K_{12r}=K_{21r}=-\omega_rR_{\eta r}\mp\sqrt{\omega_r^2R_{\eta r}^2+R_{\eta r}} \tag{51a}$$

$$K_{22r}=\mp(R_{\eta r}-2\omega_r^2R_{\eta r}^2\mp2\omega_rR_{\eta r}\sqrt{\omega_r^2R_{\eta r}^2+R_{\eta r}})^{1/2} \tag{51b}$$

$$K_{11r}=\left(\frac{1}{R_{\eta r}\omega_r^2}\mp\frac{2\sqrt{\omega_r^2R_{\eta r}^2+R_{\eta r}}}{\omega_rR_{\eta r}}-2\right)^{1/2}\sqrt{\omega_r^2R_{\eta r}^2+R_{\eta r}} \tag{51c}$$

The choice of signs is dictated by the requirement that K_r be positive definite. Hence, the elements of K_r must satisfy the inequalities

$$K_{11r}>0 \qquad K_{22r}>0 \qquad K_{11r}K_{22r}-K_{12r}^2>0 \tag{52}$$

regardless of the values of ω_r and $R_{\eta r}$. The inequality $K_{11r}>0$ is satisfied if the plus sign is chosen in Eq. (51a). Then, the second condition, $K_{22r}>0$, requires that the plus sign be chosen in Eq. (51b). The final form of the steady-state solution becomes

$$K_{12r}=K_{21r}=-\omega_rR_{\eta r}+\sqrt{\omega_r^2R_{\eta r}^2+R_{\eta r}} \tag{53a}$$

$$K_{22r}=(R_{\eta r}-2\omega_r^2R_{\eta r}^2+2\omega_rR_{\eta r}\sqrt{\omega_r^2R_{\eta r}^2+R_{\eta r}})^{1/2} \tag{53b}$$

$$K_{11r}=\left[\frac{1}{\omega_r^2}+\frac{2}{\omega_rR_{\eta r}}(\omega_r^2R_{\eta r}^2+R_{\eta r})^{3/2}-2R_{\eta r}^2\omega_r^2-R_{\eta r}\right]^{1/2} \tag{53c}$$

B. Transient Solution

The transient solution of the Riccati equation, Eq. (42), can be obtained by various methods.[8] We shall consider here the method based on the augmented matrix. For each pair ξ_r, η_r, we form the 4×4 matrix M_r as follows:

$$M_r=\begin{bmatrix}A_r^T & Q_r\\ R_r^{-1} & -A_r\end{bmatrix} \tag{54}$$

Then, the transient solution of the Riccati equation is given by

$$K_r(t_r)=E_r(t_r)L_r^{-1}(t_r) \tag{55}$$

where

$$E_r(t_r=0)=H_r \qquad L_r(t_r=0)=I \tag{56}$$

in which $t_r=t_f-t$ is the remaining time. (The subscript r denoting the remaining time should not be confused with that denoting the mode number.) This circumvents the problem of solving the nonlinear Riccati equation by solving the linear equation of twice the order

$$\begin{bmatrix}\dfrac{dE_r}{dt_r}\\[2mm] \dfrac{dL_r}{dt_r}\end{bmatrix}=M_r\begin{bmatrix}E_r(t_r)\\ L_r(t_r)\end{bmatrix} \tag{57}$$

For the gyroscopic system under consideration, we have

$$M_r=\begin{bmatrix}0 & -\omega_r & 1 & 0\\ \omega_r & 0 & 0 & 1\\ 0 & 0 & 0 & -\omega_r\\ 0 & R_{\eta r}^{-1} & \omega_r & 0\end{bmatrix} \quad (r=1,2,...,n/2) \tag{58}$$

Denoting the transition matrix of Eq. (57) by Φ and partitioning it, we can write

$$\Phi=\begin{bmatrix}\Phi_{11}(t_r) & \Phi_{12}(t_r)\\ \Phi_{21}(t_r) & \Phi_{22}(t_r)\end{bmatrix}=e^{M_rt_r} \tag{59}$$

The solution of Eq. (57) can be written in the form

$$E_r(t_r)=\Phi_{11}(t_r)E_r(t_r=0)+\Phi_{12}(t_r)L_r(t_r=0) \tag{60a}$$

$$L_r(t_r)=\Phi_{21}(t_r)E_r(t_r=0)+\Phi_{22}(t_r)L_r(t_r=0) \tag{60b}$$

Noting that the matrices E_r and L_r are 2×2, and taking the inverse of L_r and using it in Eq. (55), we finally obtain the transient solution of the Riccati equation

$$K_r(t_r)=\frac{1}{L_{r11}L_{r22}-L_{r12}L_{r21}}E_r\begin{bmatrix}L_{r22} & -L_{r12}\\ -L_{r21} & L_{r11}\end{bmatrix} \quad (r=1,...,n/2) \tag{61}$$

which is subject to the initial conditions [see Eq. (56)].

Hence, once again, in deriving the transient solution of Riccati's equation by the augmented matrix method, independent modal-space control eliminates the problem of repeatedly inverting high-order matrices. Different control methods leading to a Riccati equation in coupled form would require inversion of a large-order L matrix for every instant at which the response is desired. Numerical difficulties likely to be encountered in the approach are obvious.

C. Closed-Form Solution Corresponding to Constant Control Gains

As the last step in our optimal control problem, we shall give the closed-form solution for the optimal modal state vector w_C^* corresponding to the steady-state solution of the Riccati equation (42). We shall denote the controlled, conjugate generalized modal coordinates by the superscript C, and write the optimal gain matrix f_r^* corresponding to these coordinates in the form

$$f_r^* = \begin{bmatrix} 0 & 0 \\ f_{r21}^* & f_{r22}^* \end{bmatrix} \tag{62}$$

where we recall that the first row of f_r^* is zero because $R_{\xi r}^{-1}=0$. Inserting Eq. (44) into Eq. (34), we obtain

$$\dot{w}_r^* = A_r w_r^* + f_r^* w_r^* = (A_r + f_r^*) w_r^* \qquad (r=1,2,...,n/2) \tag{63}$$

The elements of f_r^* are obtained by using Eq. (45) in conjunction with Eqs. (53). The result is

$$f_{r21}^* = -R_{\eta r}^{-1} K_{r21} = \omega_r - \sqrt{\omega_r^2 + R_{\eta r}^{-1}} \tag{64a}$$

$$f_{r22}^* = -R_{\eta r}^{-1} K_{r22} = -[2\omega_r(-\omega_r + \sqrt{\omega_r^2 + R_{\eta r}^{-1}}) + R_{\eta r}^{-1}]^{1/2} \tag{64b}$$

Hence, Eq. (63) becomes

$$\begin{bmatrix} \dot{\xi}_r^C \\ \dot{\eta}_r^C \end{bmatrix}^* = \begin{bmatrix} 0 & \omega_r \\ -\sqrt{\omega_r^2 + R_{\eta r}^{-1}} & -[2\omega_r(-\omega_r + \sqrt{\omega_r^2 + R_{\eta r}^{-1}}) + R_{\eta r}^{-1}]^{1/2} \end{bmatrix} \begin{bmatrix} \xi_r^c \\ \eta_r^c \end{bmatrix}^* \tag{65}$$

The solution of Eq. (65), obtained by means of the Laplace transform method, has the form

$$\xi_r^*(t) = \exp[-\lambda_r\omega_r(t-t_0)]\{\xi_r^*(t_0)[\cos\omega_{dr}(t-t_0) + (\omega_r\lambda_r/\omega_{dr})\sin\omega_{dr}(t-t_0)] + \eta_r^*(t_0)(\omega_r/\omega_{dr})\sin\omega_{dr}(t-t_0)\} \tag{66a}$$

$$\eta_r^*(t) = \exp[-\lambda_r\omega_r(t-t_0)]\{\xi_r^*(t_0)[(f_{r21}^* - \omega_r)/\omega_{dr}]\sin\omega_{dr}(t-t_0) + \eta_r^*(t_0)[\cos\omega_{dr}(t-t_0) - (\omega_r\lambda_r/\omega dr)\sin\omega_{dr}(t-t_0)]\} \tag{66b}$$

where

$$\omega_{dr} = (\omega_r^2 - \omega_r f_{r21}^* - f_{r22}^*/4)^{1/2} \qquad \lambda_r = -f_{r22}^*/2\omega_r \tag{67}$$

Finally, the optimal control gain matrix corresponding to the steady-state solution is

$$F_{11}^* = \text{block-diag}\begin{bmatrix} 0 & 0 \\ \omega_r - \sqrt{\omega_r^2 + R_{\eta r}^{-1}} & -[2\omega_r(-\omega_r + \sqrt{\omega_r^2 + R_{\eta r}^{-1}}) + R_{\eta r}^{-1}]^{1/2} \end{bmatrix} \tag{68}$$

Equation (68) shows the explicit dependence of the control gains on the spacecraft natural frequencies. This dependence is obscured completely in a solution of the coupled Riccati equation. Indeed, such a solution is not a closed-form solution even in the steady-state case. Equations (66) and (68) give, for the first time, a closed-form optimal solution for a high-order dynamical system.

D. Closed-Loop Equations

The closed-loop equations, including the uncontrolled modes, can be readily obtained by considering Eqs. (24) and (25), in conjunction with Eqs. (28), (29) and (47), which yield

$$\begin{bmatrix} \dot{w}_C^* \\ \dot{w}_R \end{bmatrix} = \begin{bmatrix} A_C + F_{11}^* & 0 \\ -P_{22}^{-1}P_{21}F_{11}^* & A_R \end{bmatrix} \begin{bmatrix} w_C^* \\ w_R \end{bmatrix} \tag{69}$$

From this equation, we deduce that the eigenvalues of the closed-loop system consist of the eigenvalues of the controlled modes which have negative real parts by design, and the eigenvalues of the matrix A_R which are purely imaginary. Hence the closed-loop system, including the n uncontrolled modes, is stable. The uncontrolled modes cannot be made unstable by the optimal controls designed for the controlled modes, although they are excited by the controls, as shown by Eq. (69).

Spatial Distribution of Actuators

The equation of motion, Eq. (3), is the result of discretization of a distributed-parameter system, so that the relation between the spatial distribution of actuators and the generalized optimal control forces for the discretized system is of interest. These relations were derived in Ref. 4 in detail. However, for completeness, they will be summarized here. We shall assume that the controllers consist of torquers for the control of the rotational motions, and force actuators for the control of the elastic displacements. It is shown in Ref. 4 that θ_3 is an ignorable coordinate and does not exhibit gyroscopic behavior. Indeed, it is possible to eliminate it from the problem formulation completely, so that only the two rotations θ_1 and θ_2 appear in the following results. Let us denote the number of torquers about the x and y axes by m_a, and the number of thrusters in the z direction by k_a. It will prove convenient to introduce the notation

$$M_x = \sum_{i=1}^{m_a} M_{xi}(P_i,t) \qquad M_y = \sum_{i=1}^{m_a} M_{yi}(P_i,t) \tag{70}$$

$$F = [F_{z1} \quad F_{z2} ... F_{zka}]^T \tag{71}$$

$$y_P = [y(P_1) \quad y(P_2)...y(P_{ka})]^T \tag{72a}$$

$$x_P = [x(P_1) \quad x(P_2)...x(P_{ka})]^T \tag{72b}$$

$$B = [\Phi_z(P_1) \quad \Phi_z(P_2)...\Phi_z(P_{ka})] \tag{73}$$

where P_1, P_2,... etc., denote the positions of the controllers relative to the origin of the reference frame, and C is the moment of inertia of the complete spacecraft about the z-axis. The relationships between the components of the discretized generalized force vector Q and the actual control forces and torques are obtained in Refs. 4 and 5 in the form

$$Q^* = [M^*_x + y_P^T F^* \mid M_y^* - x_P^T F^* \mid (BF^*)^T]^T \tag{74}$$

The optimal value of the vector Q is given by Eq. (49). The actual control forces and torques must be chosen and located such that the equation is satisfied. It is shown in Ref. 5 that for uniqueness the number of actuators controlling the elastic motions must be equal to the number of elastic modes considered in the dynamical model, which means that B [see Eq. (73)] is a square matrix. Note that B must be nonsingular. There are no restrictions on the number of torquers, however. The actual control torquers and forces can be obtained from Ref. 5 in the form

$$\begin{bmatrix} M \\ F \end{bmatrix} = \left[\begin{array}{c|c} I & -DB^{-1} \\ \hline 0 & B^{-1} \end{array}\right] Q \tag{75}$$

where

$$M = [M_x \; M_y]^T \qquad D = \begin{bmatrix} y_P^T \\ -x_P^T \end{bmatrix} \tag{76}$$

Equation (76) provides clues for locating the force actuators. Indeed, they must be located such that B^{-1} exist. Recalling the generalized force vector $X(t) = [Q(t)^T \mid 0^T]^T$, and introducing the notation

$$S = \left[\begin{array}{c|c} I & -DB^{-1} \\ \hline 0 & B^{-1} \end{array}\right] \tag{77}$$

Eq. (76) yields

$$\begin{bmatrix} M \\ F \end{bmatrix} = [S \mid 0] X \tag{78}$$

Finally, using Eqs. (18) and (28) we obtain the actual controls that will yield the optimal control law given by Eq. (47) in the form

$$\begin{bmatrix} M^*(t) \\ F^*(t) \end{bmatrix} = [S \mid 0] MP \begin{bmatrix} F_{1l}^* \\ F_{2l}^* \end{bmatrix} w_C^*(t) \tag{79}$$

We observe that all the matrices involved in Eq. (79) are real, and that the required control time histories are physically realizable as soon as the vector $w_C^*(t)$ becomes available. This presents no problem because $w_C^*(t)$, or its estimate, can be obtained by means of direct measurements and/or state estimators. A discussion of this aspect of the problem can be found in Ref. 5. A block-diagram of the control scheme is shown in Fig. 1.

It is assumed by the preceding that the system is controllable. The conditions to be satisfied by the torque vector $M^*(t)$ and force vector $F^*(t)$ for the system to be controllable are given in Ref. 5.

Application to a Dual-Spin Spacecraft

The theory developed in the preceding sections has been applied to a dual-spin spacecraft (Fig. 2). Details of the dynamical formulation for such a spacecraft are given in Ref. 3. The following moments of inertia for the spacecraft and the rotor, as well as the rotor spin rate, were used: $I_x = 250.0$ kg-m^2, $I_y = 800.0$ kg-m^2, $I_z = 1200.0$ kg-m^2, $I_{x_{\text{rotor}}} = 40.0$ kg-m^2, $I_{z_{\text{rotor}}} = 200$ kg-m^2, $\Omega = 2\pi$ rad/s.

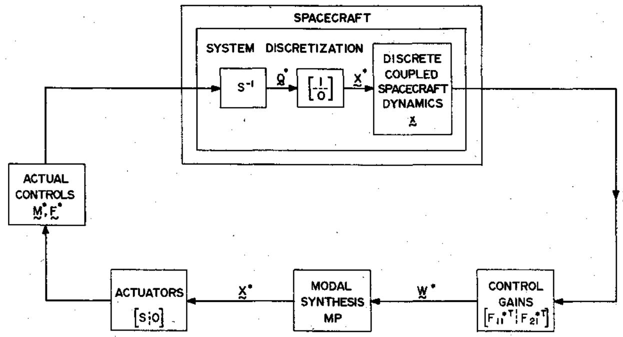

Fig. 1 Closed-loop modal-space control.

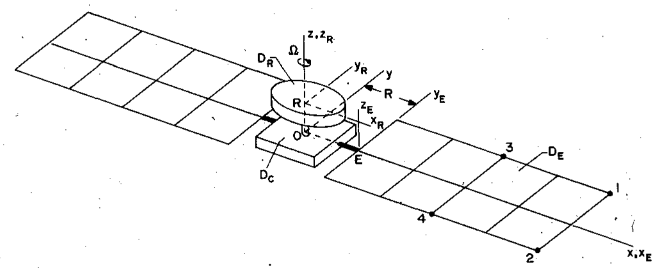

Fig. 2 The dual-spin spacecraft.

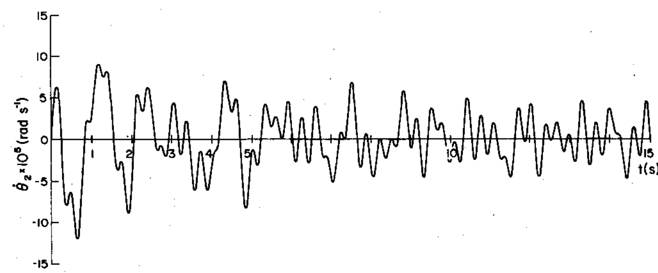

Fig. 3a Nutation rate $\dot{\theta}_2$ time history.

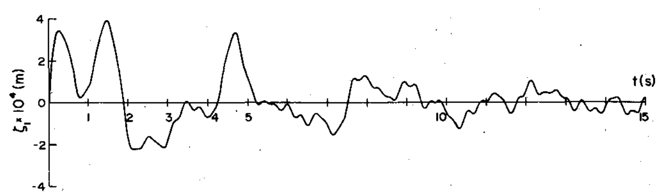

Fig. 3b First out-of-plane bending mode time history.

The in-plane motion of the elastic appendages is associated with the ignorable coordinate θ_3, which is independent of gyroscopic effects, so that it was eliminated from the problem formulation. Two nutation angles θ_1, θ_2, and four elastic coordinates ζ_1, ζ_2, ζ_3, ζ_4 were included in the dynamic model. The elastic coordinates were first out-of-plane bending, first torsional, second out-of-plane bending, and second torsional mode, respectively. Hence, the total number of generalized coordinates were six, resulting in a 12th-order dynamic model. The solution of the spacecraft eigenvalue problem by the method of Ref. 10 yielded the following spacecraft natural frequencies (in rad/s): $\omega_1 = 1.6694$, $\omega_2 = 4.0279$, $\omega_3 = 5.9125$, $\omega_4 = 8.5252$, $\omega_5 = 15.2095$, and $\omega_6 = 19.5835$. Of the twelve spacecraft modes, the first six were considered as controlled modes and the remaining six as residual. However, any other six modes could have been chosen as controlled modes. The modal matrix of the spacecraft is tabulated in Ref. 3. The spacecraft was subjected to given initial velocities, with the initial displacements assumed to be zero. The elements of the matrices R_r corresponding to the three pairs of controlled spacecraft modes were taken as $R_{\xi 1}=\infty$, $R_{\xi 2}=\infty$, $R_{\xi 3}=\infty$, $R_{\eta 1}=20.0$, $R_{\eta 2}=20.0$, and $R_{\eta 3}=20.0$. From Eq. (68), we obtained the optimal gain matrix for the controlled modes

$$F^*_{11} = \text{block-diag} \left[\begin{array}{cc|cc|cc} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.015 & -0.316 & -0.006 & -0.316 & -0.004 & -0.316 \end{array} \right]$$

The state vector of the spacecraft was obtained by using the modal transformation matrix P, i.e.,

$$x = Pw = P[w_C^{*T} \vdots w_R^T]^T$$

where the effect of the residual modes, excited by the optimal modal state vector w^*_C, is included. Figures 3a and 3b show the nutation rate $\dot{\theta}_2$ and the first out-of-plane bending coordinate ζ_1. It is seen that the residual modes did not cause any undesirable effect on the system.

The modal control law $W^*_C = F^{*T}_{11} w^*_C$ was realized by means of four thrusters ($k_a=4$) and a torquer ($m_a=1$) which applied torques about the x and y axes. The choice of four thrusters is in accordance with the four elastic coordinates considered in the dynamic simulation. Positions 1-4 in Fig. 2 show the locations of the thrusters on the elastic appendages. The control time histories of actual controllers were obtained by using Eq. (79)

$$[M^T \vdots F^T]^T = [M_x(t) M_y(t) F_1(t) F_2(t) F_3(t) F_4(t)]^T = [S \vdots 0] MP \begin{bmatrix} F^*_{11} \\ \hline F^*_{21} \end{bmatrix} w^*_C$$

$$= 10^3 \times \begin{bmatrix} -0.010 & -0.202 & 5.750 & 293.092 & -1.143 & -85.487 \\ -2.107 & -44.638 & 0.007 & 0.361 & -0.002 & -0.128 \\ 0.001 & 0.029 & 0.009 & 0.468 & -0.002 & -0.156 \\ 0.001 & 0.029 & -0.009 & -0.468 & 0.002 & 0.156 \\ -0.003 & -0.062 & -0.019 & -0.953 & 0.004 & 0.299 \\ -0.003 & -0.063 & 0.019 & 0.953 & -0.004 & -0.299 \end{bmatrix} \begin{bmatrix} \xi^*_1 \\ \eta^*_1 \\ \xi^*_2 \\ \eta^*_2 \\ \xi^*_3 \\ \eta^*_3 \end{bmatrix}$$

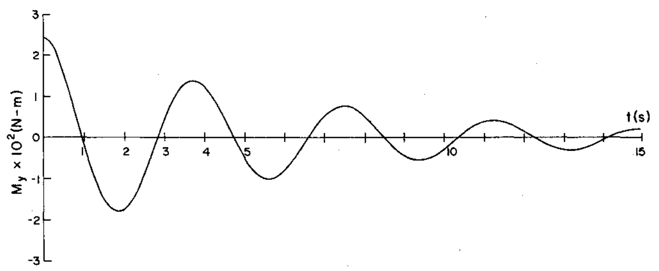

Fig. 4a Actual control torque M_y time history.

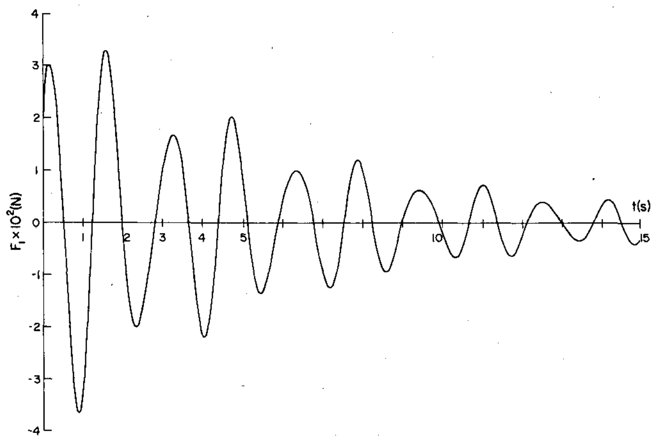

Fig. 4b Actual control force F_1 time history.

Figures 4a and 4b show the control time histories of the torquer $M_y(t)$ and the thruster in position 1 on the elastic appendage.

Conclusions

An optimal control law was presented for distributed gyroscopic systems based on the concept of *independent modal-space control.* This concept leads to block-diagonal matrix Riccati equations, each block being of second-order regardless of the order of the system to be controlled. Only decoupled modal-space control can simplify the optimal control problem to this extent, as coupled control would require the solution of a matrix Riccati equation of an order equal to the order of the system to be controlled. For very-high-order systems, such as might result from large flexible spacecraft, coupled optimal control is not feasible, which virtually leaves the proposed independent optimal modal-space control as the only alternative. By transforming to modal coordinates, the limitation on the order of a linear gyroscopic system that can be controlled optimally has been shifted from the ability to solve high-order Riccati equations to the ability to solve high-order eigenvalue problems for real symmetric matrices. A closed-form solution of the steady-state Riccati equation and augmented matrix formulation of the transient Riccati equation solution, which does not require any matrix inversion in the algorithm, are given. Note that this closed-form solution permits real-time implementation of the optimal control laws. Spatial distribution of actual controllers and their control time histories are also discussed. Numerical results are presented for a dual-spin spacecraft.

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